

# On negative flows of the AKNS hierarchy and a class of deformations of bihamiltonian structure of hydrodynamic type

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## Abstract

A deformation parameter of a bihamiltonian structure of hydrodynamic type is shown to parameterize different extensions of the AKNS hierarchy to include negative flows. This construction establishes a purely algebraic link between, on the one hand, two realizations of the first negative flow of the AKNS model and, on the other, two-component generalizations of Camassa-Holm and Dym type equations.

The two-component generalizations of Camassa-Holm and Dym type equations can be obtained from the negative order Hamiltonians constructed from the Lenard relations recursively applied on the Casimir of the first Poisson bracket of hydrodynamic type. The positive order Hamiltonians, which follow from Lenard scheme applied on the Casimir of the second Poisson bracket of hydrodynamic type, are shown to coincide with the Hamiltonians of the AKNS model. The AKNS Hamiltonians give rise to charges conserved with respect to equations of motion of two-component Camassa-Holm and two-component Dym type equations.

## 1 Introduction

Recently, the celebrated shallow-water equation obtained by Camassa and Holm [1]

$$u_t - u_{txx} = -3uu_x + 2u_xu_{xx} + uu_{xxx} - \kappa u_x, \quad \kappa = \text{const} \quad (1)$$

was extended in [2] by adding on the right hand side a term  $\rho\rho_x$  with a new variable  $\rho$ , which satisfies the continuity equation  $\rho_t + (u\rho)_x = 0$ . The model resulting from the above generalization first appeared in the study of deformations of the bihamiltonian structure of hydrodynamic type [3, 4] and was coined 2-component Camassa-Holm equation. Soon after

its derivation the model was identified with the first negative flow of the AKNS hierarchy [2].

Another well-known integrable partial differential equation of interest to our study is the Dym-type equation [5, 6, 7, 8, 9, 10]:

$$-u_{xxt} = 2u_x u_{xx} + uu_{xxx} - \kappa u_x. \quad (2)$$

It can also be extended to two-component version by adding a term  $\rho\rho_x$  on the right hand side of (2). The resulting two variable system is shown here to be equivalent to the negative flow of one of the extensions of the AKNS model. It is also equivalent to the special limiting procedure of deformations of the bihamiltonian structure of hydrodynamic type.

In this paper the following is accomplished: First, we explore the Schrödinger spectral problem of second order describing both 2-component Camassa-Holm and Dym type equation for different values of the deformation parameter  $\mu$ . We show that this Schrödinger spectral problem can be cast into the linear  $2 \times 2$  matrix spectral problem. Using  $\text{sl}(2)$  gauge invariance we transform the time evolution flow of the linear spectral problem into the AKNS first negative flow. We should point out that there exist several ways to extend the AKNS hierarchy to incorporate negative flows. These extensions are parametrized by a single parameter identified with  $\mu$ . We associate two different constructions of the negative flows of the AKNS hierarchy to the 2-component Camassa-Holm (for  $\mu = 1$ ) and Dym type equation (for  $\mu = 0$ ). The result concerning the 2-component Camassa-Holm equation constitutes an algebraic version of the proof given in [2]. Using connections of the AKNS and deformed Sinh-Gordon models to the 2-component Camassa-Holm and Dym type equations, respectively, we are able to find explicit soliton solutions given in hodographic variables. The relation to the AKNS models allows us to construct a new chain of charges conserved with respect to equations of motion of two-component Camassa-Holm and two-component Dym type equations. For both hierarchies the modified AKNS Hamiltonians provide a tower of positive order Hamiltonians obtained via the underlying Lenard relations of the Poisson brackets of hydrodynamic type from the Casimir of the second bracket.

In section 2, we briefly review the algebraic approach to the AKNS model and show how to extend the model in two different ways to negative time flows based on the zero-curvature identities. In section 3, we set up a class of two-component Schrödinger spectral problems parametrized by  $\mu$ . In the next section 4, we transform the Schrödinger spectral problem by the reciprocal transformation and linearize it. The resulting linear  $2 \times 2$  matrix spectral problem is then transformed by an  $\text{sl}(2)$  gauge transformation into the AKNS Lax spectral problem. The time flows of 2-component Camassa-Holm and Dym type equation are shown to coincide with two different negative flows of the extended AKNS model. Our construction allows us to find, in subsection 4.1, explicit soliton solutions for various values of  $\mu$ . In section 5, we reproduce equations of motion for  $\mu \neq 0$  and  $\mu = 0$  cases in the setting of deformations of the bihamiltonian structure of hydrodynamic type. Remarkably, the Hamiltonians governing positive evolution flows of the AKNS hierarchy define conserved charges for the 2-component Camassa-Holm and Dym type equations. Also the conserved charges induced by the AKNS model satisfy among themselves the Lenard relations of the bihamiltonian structure of hydrodynamic type. Thus, the Hamiltonians of the bihamiltonian structure of hydrodynamic type connected to 2-component Camassa-Holm and Dym type

equations split into two chains, one of the positive order induced by the AKNS hierarchy and one of the negative order containing generators of the equations of motion defining both hierarchies.

## 2 Extended AKNS model

First, let us present the AKNS hierarchy in the setting of the  $\text{sl}(2)$  loop algebra endowed with homogeneous gradation defined by the operator  $\lambda d/d\lambda$ . A variable  $\lambda$  plays a double role of a loop parameter of the loop algebra and a spectral parameter of the underlying hierarchy. The matrix Lax operator  $L$  for the AKNS hierarchy reads:

$$L = \frac{\partial}{\partial y} - \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} - \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}, \quad (3)$$

where  $\partial/\partial y$  is the derivative with respect to “space” variable  $y$ . The matrix Lax operator can be compactly written as  $L = \partial/\partial y - E - A_0$ , with  $E = \lambda\sigma_3$  and the matrix  $A_0 = q\sigma_+ + r\sigma_-$ , where  $\sigma_3$  is the Pauli matrix and  $\sigma_\pm$  are given in terms of other Pauli matrices  $\sigma_1, \sigma_2$

$$\sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We work within an algebraic approach to the integrable models based on the linear spectral problem  $L(\Psi) = 0$ , which simplifies considerably under a dressing transformation:

$$\Theta^{-1} \left( \frac{\partial}{\partial y} - E - A_0 \right) \Theta = \frac{\partial}{\partial y} - E, \quad (4)$$

where the dressing matrix  $\Theta = \exp(\sum_{i<0} \lambda^i \theta^{(i)})$  is an exponential in negative powers of the spectral parameter  $\lambda$  on a formal loop space of  $\text{sl}(2)$ . Similarly, for higher flows we obtain

$$\Theta^{-1} \left( \frac{\partial}{\partial t_n} - E^{(n)} - \sum_{i=0}^{n-1} \lambda^i D_n^{(i)} \right) \Theta = \frac{\partial}{\partial t_n} - E^{(n)}, \quad (5)$$

where  $E^{(n)} = \lambda^n \sigma_3$  and terms  $D_n^{(i)}$  are obtained from projection  $(\Theta E^{(n)} \Theta^{-1})_+$  of  $\Theta E^{(n)} \Theta^{-1}$  on the positive powers of  $\lambda$  via expansion relation:

$$(\Theta E^{(n)} \Theta^{-1})_+ = E^{(n)} + \sum_{i=0}^{n-1} \lambda^i D_n^{(i)}.$$

These dressing relations give rise to the zero-curvature conditions for the positive flows of the AKNS hierarchy

$$\left[ \frac{\partial}{\partial y} - E - A_0, \frac{\partial}{\partial t_n} - E^{(n)} - \sum_{i=0}^{n-1} D_n^{(i)} \right] = \Theta \left[ \frac{\partial}{\partial y} - E, \frac{\partial}{\partial t_n} - E^{(n)} \right] \Theta^{-1} = 0. \quad (6)$$

In particular, for  $n = 2$  we obtain the second flow of the AKNS hierarchy:

$$\frac{\partial r}{\partial t_2} = -\frac{1}{2}r_{yy} + qr^2 \quad ; \quad \frac{\partial q}{\partial t_2} = \frac{1}{2}q_{yy} - q^2r, \quad (7)$$

which reproduces the familiar vector non-linear Schrödinger equation.

According to [11], the Hamiltonian densities of the AKNS model are defined as

$$\mathcal{H}_n = -\text{tr}(E^{(0)}A^{(-n)}) = \frac{1}{2}\sum_{k=0}^{n-1}\text{tr}(A^{(-k)}A^{(1+k-n)}), \quad (8)$$

where  $A^{(-n)}$  are given by

$$\Theta_y\Theta^{-1} = \sum_{k=1}^{\infty}A^{(-k)}\lambda^{-k},$$

where the symbol  $\text{tr}$  in expression (8) denotes a  $\text{sl}(2)$  trace. We list the first two Hamiltonians. Inserting  $n = 1$  in (8) we obtain:

$$\mathcal{H}_1 = -\text{tr}(E^{(0)}A^{(-1)}) = \frac{1}{2}\text{tr}(A_0^2) = rq. \quad (9)$$

Similarly, for  $n = 2$  we obtain

$$\mathcal{H}_2 = qr_y. \quad (10)$$

Next, we extend the AKNS model to include negative grade time evolution equations governed by the zero-curvature equations [12]

$$\left[\frac{\partial}{\partial y} - E - A_0, \frac{\partial}{\partial t_{-n}} - D^{(-1)} - D^{(-2)} - \dots - D^{(-n)}\right] = 0. \quad (11)$$

Here, we only consider the first negative flow with  $n = 1$  and set for brevity  $s = t_{-1}$ . In this case the compatibility equation (11) reduces to

$$(A_0)_s - D_y^{(-1)} + [E + A_0, D^{(-1)}] = 0. \quad (12)$$

A general solution of the compatibility equation (12) is given by

$$D^{(-1)} = B\mathcal{E}^{(-1)}B^{-1}, \quad A_0 = B_y B^{-1}, \quad (13)$$

in terms of the zero-grade group element,  $B$ , of  $\text{SL}(2)$ , that satisfies equation:

$$(B_y B^{-1})_s = [B\mathcal{E}^{(-1)}B^{-1}, E] \quad (14)$$

or, equivalently,

$$(B^{-1}B_s)_y = [\mathcal{E}^{(-1)}, B^{-1}EB]. \quad (15)$$

Here  $\mathcal{E}^{(-1)}$  is an element of  $\text{sl}(2)$  algebra of  $-1$  grade.

Remarkably, the compatibility of the  $t_{-1}$  flow with positive  $t_n$ ,  $n \geq 1$  flows does not require that the matrix  $\mathcal{E}^{(-1)}$  commutes with  $E = \lambda\sigma_3$ , as pointed out in [13] and [14]. It

turns out that all possible cases are parametrized by a parameter  $\mu$  and fall into two main classes depending on whether  $\mu$  takes non-zero or zero value. The corresponding generic choices of  $\mathcal{E}^{(-1)}$  are:

$$\mathcal{E}^{(-1)} = \begin{cases} \mu \sigma_3 / 4\lambda & \text{for } \mu \neq 0 \\ \sigma_+ / \lambda & \text{for } \mu = 0. \end{cases} \quad (16)$$

Note that the value of determinant of  $\mathcal{E}^{(-1)}$  is equal to  $-\mu^2 / 16\lambda^2$  and 0, respectively. There exist other choices of  $\mathcal{E}^{(-1)}$  for these values of the determinant but they only lead to the gauge equivalent copies of hierarchies derived from the choice (16).

### 3 A class of two-component Schrödinger spectral problems

Consider a linear system

$$\psi_{xx} = \left( \frac{\mu^2}{4} - m\lambda + \rho^2\lambda^2 \right) \psi, \quad (17)$$

$$\psi_t = -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}u_x\psi, \quad (18)$$

for some arbitrary constant  $\mu$  (see [3, 2] for  $\mu = 1$  and [15] for  $\mu = 0$ ). Compatibility condition for the above system, yields three independent equations

$$\rho_t = -(u\rho)_x, \quad (19)$$

$$m_t = -2mu_x - m_xu + \rho\rho_x, \quad (20)$$

$$m_x = \mu^2u_x - u_{xxx}, \quad (21)$$

corresponding to coefficients of  $\lambda^2$ ,  $\lambda$  and  $\lambda^0$  in the expansion of  $\psi_{xxt} - \psi_{txx} = 0$ . Equation (21) can be integrated to yield:

$$m = \mu^2u - u_{xx} + \frac{1}{2}\kappa. \quad (22)$$

Here,  $\kappa$  is an integration constant. For  $\mu \neq 0$  that integration constant can be removed by transforming the system by Galilean transformation:

$$x' = x + vt, \quad t' = t, \quad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} = v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}.$$

In the primed system equation (20) becomes:

$$vm_{x'} + m_{t'} = -2mu_{x'} - m_{x'}u + \rho\rho_{x'} = -2\left(\mu^2u - u_{x'x'} + \frac{1}{2}\kappa\right)u_{x'} - m_{x'}u + \rho\rho_{x'}$$

Next, performing a shift  $u \rightarrow u - v$  and choosing velocity  $v$  such that  $v = \kappa/2\mu^2$  eliminates the linear terms in  $u_x$  and  $u_{xxx}$  from the above equation. Clearly, the above argument works only for  $\mu \neq 0$  and from now on we put the integration constant  $\kappa$  to zero as long as  $\mu \neq 0$ .

Note, that the positive constant  $\mu^2$ , that is different from one can be absorbed by appropriately redefining fields and derivatives. Defining  $\tilde{\lambda} = \lambda/\mu^2$ ,  $\tilde{\rho} = \rho\mu$ ,  $\tilde{u} = u\mu^2$  and new variables  $\tilde{x}$  and  $\tilde{t}$  such that  $\partial_x = \mu\partial_{\tilde{x}}$ ,  $\partial_t = (1/\mu)\partial_{\tilde{t}}$  allows us to rewrite a linear system (17)-(18) as

$$\psi_{\tilde{x}\tilde{x}} = \left( \frac{1}{4} - m\tilde{\lambda} + \tilde{\rho}^2\tilde{\lambda}^2 \right) \psi, \quad (23)$$

$$\psi_{\tilde{t}} = -\left(\frac{1}{2\tilde{\lambda}} + \tilde{u}\right)\psi_{\tilde{x}} + \frac{1}{2}\tilde{u}_{\tilde{x}}\psi. \quad (24)$$

with  $m = \mu^2u - u_{xx} + \kappa/2 = \tilde{u} - \tilde{u}_{\tilde{x}\tilde{x}} + \kappa/2$ . Thus, for  $\mu^2 \neq 1$  the spectral system has been transformed to the canonical system with  $\mu^2 = 1$ .

In case of a negative  $\mu^2$  (imaginary  $\mu$ ) we make the changes as above but with  $|\mu^2|$  instead of  $\mu^2$  and arrive at

$$\psi_{\tilde{x}\tilde{x}} = \left( -\frac{1}{4} - m\tilde{\lambda} + \tilde{\rho}^2\tilde{\lambda}^2 \right) \psi. \quad (25)$$

Thus, only three cases of  $\mu^2 = -1, 0, 1$  need to be considered separately, as concerns equations of motion.

The case of  $\mu^2 = 1$  corresponds to the 2-component Camassa-Holm model with  $m = u - u_{xx} + \kappa/2$ , introduced in [3, 2], while the case of  $\mu^2 = -1$  corresponds to equation (20) with  $m = -u - u_{xx} + \kappa/2$ , which for  $\rho = 0$  is well-known to possess the compacton solutions [16].

For  $\mu = 0$ , we obtain from (22)  $m = -u_{xx} + \kappa/2$ . Inserting this value of  $m$  into equation (20) yields

$$u_{xxt} = -2u_xu_{xx} - uu_{xxx} + \kappa u_x - \rho\rho_x. \quad (26)$$

For  $\rho = 0$  this is the Dym type equation (2).

After one integration (and ignoring the integration constant) we obtain from (26)

$$\begin{aligned} 0 &= u_{xt} + uu_{xx} - \kappa u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \\ &= (u_t + uu_x)_x - \kappa u + \frac{1}{2}(-u_x^2 + \rho^2). \end{aligned} \quad (27)$$

In terms of a new function

$$v = \frac{1}{2}(-u_x^2 + \rho^2) \quad (28)$$

we can cast equation (27) in the following form

$$(u_t + uu_x)_x - \kappa u + v = 0. \quad (29)$$

In addition, it follows from equations (27) and (19) that  $v$  defined by relation (28) also satisfies

$$v_t + (u(v + u\kappa/2))_x = 0, \quad (30)$$

which becomes a continuity equation in the  $\kappa = 0$  limit. The linear system corresponding to equations (29) and (30) takes a form

$$\begin{aligned}\psi_{xx} &= \left( (u_{xx} - \kappa/2) \lambda + (2v + u_x^2) \lambda^2 \right) \psi, \\ \psi_t &= -\left( \frac{1}{2\lambda} + u \right) \psi_x + \frac{1}{2} u_x \psi.\end{aligned}\tag{31}$$

## 4 Transformation to the first order spectral problem. Algebraic Connection to the AKNS model

Now, for an arbitrary  $\mu$  we perform a reciprocal transformation  $(x, t) \mapsto (y, s)$  defined by relations

$$dy = \rho dx - \rho u dt, \quad ds = dt, \tag{32}$$

and

$$\frac{\partial}{\partial x} = \rho \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} - \rho u \frac{\partial}{\partial y}. \tag{33}$$

The commutativity of derivatives with respect to  $s$  and  $y$  variables is ensured by the continuity equation (19). Applying the reciprocal transformation and then redefining  $\psi$  by  $\varphi = \sqrt{\rho} \psi$  as in [2] leads from the spectral problem (17)-(18) to :

$$\varphi_{yy} = (\lambda^2 - P \lambda - Q) \varphi, \tag{34}$$

$$\varphi_s = -\frac{\rho}{2\lambda} \varphi_y + \frac{\rho_y}{4\lambda} \varphi, \tag{35}$$

where

$$\begin{aligned}P &= \frac{m}{\rho^2} \\ Q &= -\frac{\mu^2}{4\rho^2} - \frac{\rho_{yy}}{2\rho} + \frac{\rho_y^2}{4\rho^2}.\end{aligned}\tag{36}$$

Our main point in this section is that we can rewrite the second-order spectral problem (34)-(35) as a first-order linear problem:

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix}_y = A \begin{bmatrix} \varphi \\ \eta \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix}_s = D \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \tag{38}$$

involving sl(2) matrices:

$$\begin{aligned}A &= \begin{bmatrix} g & \lambda \\ \lambda - P & -g \end{bmatrix} = \lambda \sigma_1 + g \sigma_3 - P \sigma_-, \\ D &= \frac{1}{\lambda} D_0 - \frac{1}{2} \rho \sigma_1, \quad D_0 = -\frac{\mu}{4} \sigma_3 + \frac{1}{4} (P \rho - 2g_s) \sigma_-.\end{aligned}\tag{39}$$

Note that determinant of  $D_0$  is equal to  $\det D_0 = -\mu^2/16$  and, therefore, the matrix  $D_0$  becomes singular for  $\mu = 0$ . Eliminating  $\eta$  from the linear spectral problem (37)-(38) reproduces equations (34)-(35) for  $\varphi$  providing that function  $g(y, s)$  appearing in (39) satisfies the Riccati equation

$$Q = -g^2 - g_y$$

for  $Q$  given in equation (36). Remarkably, the solution to the above Riccati equation takes a local form of

$$g(y, s) = \frac{\mu}{2\rho} + \frac{\rho_y}{2\rho}. \quad (40)$$

The zero-curvature equation

$$A_s - D_y + [A, D] = 0, \quad (41)$$

can easily be derived from the linear spectral problem (37)-(38). It is equivalent to equations:

$$P_s = \rho_y, \quad Q_s + \frac{1}{2}P_y\rho + P\rho_y = 0. \quad (42)$$

These equations were found in [2] directly from compatibility of equations (34)-(35). It follows from the first of the above equations that there exists a function  $f(y, s)$  such that  $P = f_y$  and  $\rho = f_s$ .

By plugging  $\rho = f_s$  and  $P = f_y$  into the second relation in (42) one obtains as in [2]:

$$\mu^2 \frac{f_{ss}}{2f_s^3} + f_{sy}f_y + \frac{1}{2}f_sf_{yy} - \frac{f_{ssyy}}{2f_s} + \frac{f_{ssy}f_{sy}}{2f_s^2} + \frac{f_{ss}f_{syy}}{2f_s^2} - \frac{f_{ss}f_{sy}^2}{2f_s^3} = 0. \quad (43)$$

This appears to be the only condition, which the function  $f$  has to satisfy in order to be a solution of the model.

For  $\mu = 0$  equation (43) simplifies to

$$\left( f_s^2 f_y - f_{ssy} + \frac{f_{ss}f_{sy}}{f_s} \right)_y = 0. \quad (44)$$

Integrating the above equation once and setting the integration constant to  $\kappa/2$  (see explanation below) yields

$$\frac{f_{ssy}}{f_s^2} - \frac{f_{ss}f_{sy}}{f_s^3} + \frac{\frac{1}{2}\kappa}{f_s^2} = f_y, \quad (45)$$

or

$$(\ln f_s)_{sy} + \frac{1}{2}\kappa/f_s = f_s f_y.$$

Indeed, multiplying both sides of equation (45) by  $f_s^2$  and taking a derivative with respect to  $y$  yields (44). It remains to be shown that the choice of  $\kappa/2$  as the integration constant in equation (45) was consistent with equations of motion. To do this we start by recalling that  $P = m/\rho^2$  with

$$m = -u_{xx} + \frac{1}{2}\kappa = -\rho(\rho u_y)_y + \frac{1}{2}\kappa$$

in the  $\mu = 0$  case. The continuity equation (19) reads in the hodographic variables  $\rho_s = -\rho^2 u_y$ . Accordingly, substituting  $u_y = (1/\rho)_s$  into  $P$  we get

$$P = \frac{(\rho_s/\rho)_y}{\rho} + \frac{\frac{1}{2}\kappa}{\rho^2} = f_y,$$

which is precisely equation (45).

Let us turn our attention back to the zero-curvature equation (41). This equation is invariant under the  $\text{sl}(2)$  gauge transformation:

$$A \longrightarrow UAU^{-1} + U_y U^{-1}, \quad D \longrightarrow UDU^{-1} + U_s U^{-1}.$$

This invariance will be used in what follows to cast the linear spectral problem (37)-(38) in the standard form of the first positive and first negative flow equations of the  $\text{sl}(2)$  AKNS hierarchy.

As a first step we gauge away the term  $-\frac{1}{2}\rho\sigma_1$  of order  $\lambda^0$  in the expression for  $D$  in equation (39) by choosing

$$U = \exp\left(\frac{1}{2}f(y, s)\sigma_1\right) = \cosh\frac{f}{2} + \sigma_1 \sinh\frac{f}{2}$$

so that  $U_s U^{-1} - \frac{1}{2}\rho\sigma_1 = 0$ , due to  $f_s = \rho$ . Consequently,

$$\begin{aligned} A &\rightarrow UAU^{-1} + U_y U^{-1} = U(\lambda\sigma_1 + g\sigma_3 - P\sigma_-)U^{-1} + \frac{1}{2}P\sigma_1 \\ &= \lambda\sigma_1 + \sigma_3 \left( g \cosh f - \frac{1}{2}P \sinh f \right) - i\sigma_2 \left( g \sinh f - \frac{1}{2}P \cosh f \right) \\ D &\rightarrow \frac{1}{\lambda}UD_0U^{-1} = \frac{1}{4\lambda} [(-\sigma_1 \cosh f - i\sigma_2 \sinh f)\mu \\ &\quad + (P\rho - 2g_s)(\sigma_3 + \sigma_1 \cosh f + i\sigma_2 \sinh f)] \end{aligned}$$

Note that the gauge transformed of the matrix  $D$  is now proportional to  $1/\lambda$ .

Next, we define the constant matrix  $\Omega = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$ , that by a similarity transformation maps  $\sigma_1$  to  $\sigma_3$ ,  $\Omega\sigma_1\Omega^{-1} = -\sigma_3$ . Note also that  $\Omega\sigma_2\Omega^{-1} = -\sigma_2$ . The combined gauge transformations first by  $U$  and then by  $\Omega$  produce the final result

$$\begin{aligned} A &\rightarrow E + A_0 = \Omega [UAU^{-1} + U_y U^{-1}] \Omega^{-1} \\ &= \lambda\sigma_3 + \sigma_1 \left( g \cosh f - \frac{1}{2}P \sinh f \right) + i\sigma_2 \left( g \sinh f - \frac{1}{2}P \cosh f \right) \\ D &\rightarrow D^{(-1)} = \frac{1}{\lambda}\Omega UD_0U^{-1}\Omega^{-1} = \frac{1}{4\lambda} \left[ (P\rho - 2g_s)\sigma_3 \right. \\ &\quad \left. + \sigma_1 \left( (P\rho - 2g_s) \sinh f - \mu \cosh f \right) + i\sigma_2 \left( (P\rho - 2g_s) \cosh f - \mu \sinh f \right) \right] \end{aligned} \tag{46}$$

In the above, we re-introduced  $E = \lambda\sigma_3$  and  $A_0 = r\sigma_- + q\sigma_+$ . Comparing with the right hand side of equation (46) we find that

$$q = e^f \left( g - \frac{P}{2} \right), \quad r = e^{-f} \left( g + \frac{P}{2} \right). \tag{47}$$

Furthermore, defining matrix entries of  $D^{(-1)}$  as

$$D^{(-1)} = \frac{1}{\lambda} \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}, \quad (48)$$

we find from (46), that  $\alpha, \beta$  and  $\gamma$  are given by

$$\begin{aligned} \alpha &= \frac{1}{4} (P\rho - 2g_s) \\ \beta &= e^f \left( \alpha - \frac{\mu}{4} \right) \\ \gamma &= -e^{-f} \left( \alpha + \frac{\mu}{4} \right). \end{aligned} \quad (49)$$

They satisfy the determinant formula  $\alpha^2 + \beta\gamma = -\mu^2/16$ . The matrix entries of  $A_0$  and  $D^{(-1)}$  enter the following simple identities

$$2\alpha = \beta e^{-f} - \gamma e^f, \quad -\frac{\mu}{2} = \beta e^{-f} + \gamma e^f \quad (50)$$

$$P = re^f - qe^{-f}, \quad g = \frac{1}{2} (re^f + qe^{-f}). \quad (51)$$

It follows that the linear spectral problem (37)-(38) has been transformed by the above gauge transformation to:

$$\Psi_y = (E + A_0) \Psi = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \Psi + \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} \Psi \quad (52)$$

$$\Psi_s = D^{(-1)} \Psi = \frac{1}{\lambda} \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \Psi \quad (53)$$

for some two-component object  $\Psi$ .

We recognize in (52) the spectral problem  $L(\Psi) = 0$  with the AKNS Lax operator given by equation (3). It also follows easily that equation (12) is the compatibility equation of the spectral equations (52)-(53). The compatibility equation (12) yields

$$q_s = -2\beta, \quad r_s = 2\gamma. \quad (54)$$

when projected on zero grade, and

$$\begin{aligned} \alpha_y &= \frac{1}{2}(rq)_s = q\gamma - r\beta \\ \beta_y &= -2\alpha q \\ \gamma_y &= 2\alpha r, \end{aligned} \quad (55)$$

when projected on  $-1$  grade. Equations (55) can also be directly derived from equations of motion (42).

## 4.1 Examples and Solutions

Let us recall that a general solution of the compatibility equation (12) is given by expressions from equation (13). It is convenient to parametrize the  $SL(2)$  group element  $B$  appearing in expressions in (13) by the  $sl(2)$  algebra elements through the Gauss decomposition:

$$B = e^{\chi\sigma_-} e^{R\sigma_3} e^{\phi\sigma_+}. \quad (56)$$

### 4.1.1 The case of $\mu \neq 0$

As an example, we first consider  $\mu^2 = 4$  with  $\mathcal{E}^{(-1)} = \sigma_3/2\lambda$  according to equation (16). As in [12] in order to match the number of independent modes in the matrix  $A_0$  we impose two “diagonal” constraints  $\text{Tr}(B_y B^{-1} \sigma_3) = 0$  and  $\text{Tr}(B^{-1} B_s \sigma_3) = 0$ , which effectively eliminate  $R$  in terms of  $\phi$  and  $\chi$ .

From  $B_y B^{-1} = q\sigma_+ + r\sigma_-$  we obtain the following representation for  $q$  and  $r$ :

$$q = -\frac{h_y}{\Delta} e^R \quad ; \quad r = -\bar{h}_y e^{-R} \quad (57)$$

where

$$h = \phi e^R \quad ; \quad \bar{h} = \chi e^R \quad ; \quad \Delta = 1 + h \bar{h} \quad (58)$$

with a non-local field  $R$  being determined in terms  $h$  and  $\bar{h}$  from the “diagonal” constraints:

$$\text{Tr}(B_y B^{-1} \sigma_3) = 0 \rightarrow R_y = \frac{\bar{h} h_y}{\Delta}, \quad (59)$$

$$\text{Tr}(B^{-1} B_s \sigma_3) = 0 \rightarrow R_s = \frac{h \bar{h}_s}{\Delta}. \quad (60)$$

The zero curvature equations are in this parameterization

$$q_s = \left( -\frac{h_y}{\Delta} e^R \right)_s = -2h e^R, \quad (61)$$

$$r_s = \left( -\bar{h}_y e^{-R} \right)_s = -2\bar{h} \Delta e^{-R}. \quad (62)$$

The two-parameter solution to the above equations can be deduced from a method combining dressing and vertex techniques as described e.g. in [17]. The explicit expression is found to be given by:

$$\begin{aligned} h &= \frac{b \exp(s/\gamma_2 - y\gamma_2)}{1 + \Gamma \exp\left(s\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right) - y(\gamma_2 - \gamma_1)\right)} \\ \bar{h} &= \frac{a \exp(-s/\gamma_1 + y\gamma_1)}{1 + \Gamma \exp\left(s\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right) - y(\gamma_2 - \gamma_1)\right)} \\ e^R &= \frac{1 + \frac{\gamma_1}{\gamma_2} \Gamma \exp\left(s\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right) - y(\gamma_2 - \gamma_1)\right)}{1 + \Gamma \exp\left(s\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right) - y(\gamma_2 - \gamma_1)\right)}, \end{aligned} \quad (63)$$

where

$$\Gamma = \frac{ab\gamma_1\gamma_2}{(\gamma_1 - \gamma_2)^2}$$

is given in terms of four arbitrary constants  $a, b, \gamma_1, \gamma_2$ . Higher multi-soliton solutions can be obtained using the same straightforward procedure.

Comparing with equations (54) we find that

$$he^R = e^f \alpha_-, \quad \bar{h}\Delta e^{-R} = e^{-f} \alpha_+ \quad (64)$$

where we introduced the notation

$$\alpha_{\pm} = \alpha \pm \frac{1}{2}.$$

By multiplying the above two relations we find that

$$h\bar{h}\Delta = (\Delta - 1)\Delta = \alpha_+ \alpha_- = \alpha^2 - \frac{1}{4}.$$

Solutions to this quadratic equation are

$$\Delta = \pm \alpha_{\pm}, \quad h\bar{h} = \Delta - 1 = \pm \alpha_{\mp}. \quad (65)$$

Adding two relations in (64) we get

$$2\alpha = \alpha_+ + \alpha_- = hx + \frac{1}{x}\bar{h}\Delta, \quad x = e^{R-f}.$$

Solving this quadratic equation yields:

$$f_{\pm} = R + \ln h - \ln \alpha_{\pm}. \quad (66)$$

Equation (65) contains two solutions. The first one, namely,  $\alpha_+ = \Delta$ ,  $\alpha_- = \Delta - 1$ , when inserted into equation (66) yields

$$f_+ = R + \ln h - \ln \Delta, \quad f_- = R + \ln h - \ln(\Delta - 1) = R - \ln \bar{h}.$$

while the second solution,  $\alpha_+ = -(\Delta - 1)$ ,  $\alpha_- = -\Delta$ , leads to

$$f_+ = R + \ln h - \ln(-\Delta + 1), \quad f_- = R + \ln h - \ln(-\Delta).$$

Thus, the Bäcklund transformation

$$f_+ = f_- + \epsilon \ln \left( \frac{\Delta}{\Delta - 1} \right), \quad \epsilon = \pm 1$$

relates the two values  $f_+$  and  $f_-$ . In the reduced case of sinh-Gordon equation with  $h = \bar{h}$  we find that  $\Delta = 1 + h^2$  and

$$R = \frac{1}{2} \ln(1 + h^2) = \frac{1}{2} \ln \Delta, \quad \ln(\Delta - 1) = 2 \ln h.$$

It follows that all the above values of  $f_+, f_-$  can be summarized as:

$$f_{\epsilon} = \frac{\epsilon}{2} \ln \frac{1 + h^2}{h^2}, \quad \epsilon = \pm 1.$$

#### 4.1.2 The case of $\mu = 0$

In the case of  $\mu = 0$ , the matrix  $D^{(-1)}$  from equation (48) takes a simple form

$$D^{(-1)} = \frac{\alpha}{\lambda} \begin{bmatrix} 1 & e^f \\ -e^{-f} & -1 \end{bmatrix}, \quad (67)$$

which according to definition (16) is reproduced by

$$\frac{1}{\lambda} B \sigma_+ B^{-1} = \frac{1}{\lambda} \begin{bmatrix} -\chi & 1 \\ -\chi^2 & \chi \end{bmatrix} e^{2R}$$

for

$$\begin{aligned} \chi &= -e^{-f} \\ e^{2R} &= \alpha e^f \end{aligned} \quad (68)$$

or  $R = (f + \ln \alpha)/2$ .

From equation (55) we find for  $\mu = 0$  that  $\alpha_y = -2\alpha g$ . Therefore

$$g = -\frac{\alpha_y}{2\alpha} = -\frac{1}{2}(\ln \alpha)_y \quad (69)$$

and by comparing with definition (40) we conclude that

$$(\rho\alpha)_y = 0. \quad (70)$$

Next, we calculate

$$B_y B^{-1} = (R_y - \chi \phi_y e^{2R}) \sigma_3 + \phi_y e^{2R} \sigma_+ + (\chi_y + 2\chi R_y - \chi^2 \phi_y e^{2R}) \sigma_- . \quad (71)$$

Imposing condition  $\text{Tr}(B_y B^{-1} \sigma_3) = 0$  implies  $R_y - \chi \phi_y e^{2R} = 0$  or

$$\phi_y = -R_y/\alpha = -\frac{1}{2\alpha} \left( f_y + \frac{\alpha_y}{\alpha} \right) = \frac{1}{\alpha} (g - P/2) .$$

What remains of expression (71) is now given by

$$B_y B^{-1} = \phi_y e^{2R} \sigma_+ + (\chi_y + \chi R_y) \sigma_- = (R_y/\chi) \sigma_+ + (\chi_y + \chi R_y) \sigma_- .$$

Recalling relations (68) and (69) we obtain the desired results

$$q = \phi_y e^{2R} = \left( g - \frac{1}{2}P \right) e^f, \quad r = \chi_y + \chi R_y = \left( g + \frac{1}{2}P \right) e^{-f} ,$$

that reproduce expressions (47).

The compatibility equation

$$(B_y B^{-1})_s = [B \sigma_+ B^{-1}, \sigma_3] = \begin{bmatrix} 0 & -2 \\ -2\chi^2 & 0 \end{bmatrix} e^{2R}$$

yields the following equations of motion

$$\left( \frac{R_y}{\chi} \right)_s = -2e^{2R} \quad (72)$$

$$\chi_{ys} + 2\chi_s R_y = 0 \rightarrow (\chi_s e^{2R})_y = 0. \quad (73)$$

Equation (73) implies that

$$\chi_s = c_3(s) e^{-2R}, \quad (74)$$

where  $c_3(s)$  is an arbitrary function of  $s$  only.

From equations (68) we find that

$$\chi_s = f_s e^{-f} = \rho e^{-f} = c_3(s) \alpha^{-1} e^{-f}$$

and therefore  $\rho = c_3(s)/\alpha$  in agreement with relation (70). It follows that  $c_3(s)$  has to be different from zero for consistency of the model with  $\rho \neq 0$ .

Integrating relations (74) and (72) leads to

$$\chi = \int^s c_3(s) e^{-2R} ds + c_2(y) \quad (75)$$

$$\frac{R_y}{\chi} = -2 \int^s e^{2R} ds + c_1(y), \quad (76)$$

where  $c_1, c_2$  depend at most on  $y$ . Combining these two equations and setting  $c_3(s)$  to be a constant  $c_3$  we get the deformed sinh-Gordon equation for  $R$  [18]:

$$R_y = -2c_3 \int^s e^{-2R} ds \int^s e^{2R} ds - 2c_2 \int^s e^{2R} ds + c_1 c_3 \int^s e^{-2R} ds + c_1 c_2 \quad (77)$$

or

$$R_{ys} = -2c_3 \left( \int^s e^{-2R} ds \int^s e^{2R} ds \right)_s - 2c_2 e^{2R} + c_1 c_3 e^{-2R}.$$

The one soliton solution to the above equation with  $c_1 = c_2 = 0$  is given by (see also [18])

$$R(s, y) = \frac{s}{2} + 2c_3 y + \ln \left( \frac{k_0 e^{k_1 s + k_2 y} + k_1 + 1}{k_0 e^{k_1 s + k_2 y} - k_1 + 1} \right), \quad (78)$$

where

$$k_2 = 8c_3 \frac{k_1}{k_1^2 - 1}$$

and  $k_0, k_1$  are real constants. The corresponding expression for  $\chi$  is

$$\chi(s, y) = -e^{-f} = -e^{-s-4c_3 y} \frac{(k_1 - 1)^2/(k_1 + 1) + k_0 e^{k_1 s + k_2 y}}{k_0 e^{k_1 s + k_2 y} + k_1 + 1} c_3. \quad (79)$$

The above function  $\chi$  together with  $R$  from (78) solve equations (72)-(73). The function  $f$  defined by equation (79) provides a one-soliton solution to equation (44). It satisfies

$$f_s = \frac{c_3}{\alpha}$$

with

$$\alpha = \frac{((k_1 - 1)^2 / (k_1 + 1) + k_0 e^{k_1 s + k_2 y}) (k_0 e^{k_1 s + k_2 y} + k_1 + 1)}{(k_0 e^{k_1 s + k_2 y} - k_1 + 1)^2} c_3. \quad (80)$$

Eliminating  $\alpha$  from equation (72) using (68) we get

$$-(\ln f_s)_{ys} + f_s f_y = 4\alpha = \frac{4c_3}{f_s}.$$

Therefore, comparing with (45), we see that  $\kappa = 8c_3$  and  $f$  given in (79) satisfies (45) and therefore also (44). One-soliton solution  $R(s, y)$  given by expression (78) satisfies therefore:

$$R_y = -\frac{\kappa}{4} \int^s e^{-2R} ds \int^s e^{2R} ds.$$

## 5 Bihamiltonian structure

### 5.1 Bihamiltonian structure of the 2-component Camassa-Holm model

As in [3], we consider the following bihamiltonian structure:

$$\begin{aligned} \{w_1(x), w_1(x')\}_1 &= \{w_2(x), w_2(x')\}_1 = 0, \\ \{w_1(x), w_2(x')\}_1 &= \delta'(x - x') - \frac{1}{\mu} \delta''(x - x'). \\ \{w_1(x), w_1(x')\}_2 &= 2\delta'(x - x'), \\ \{w_1(x), w_2(x')\}_2 &= w_1(x)\delta'(x - x') + w'_1(x)\delta(x - x'), \\ \{w_2(x), w_2(x')\}_2 &= w_2(x)\delta'(x - x') + \partial_x(w_2(x)\delta(x - x')), \end{aligned} \quad (81)$$

where  $1/\mu$  now plays a role of the deformation parameter. There exists an hierarchy of Hamiltonians related through Lenard relations [3]

$$\{w_i(x), H_{-j}\}_2 = j\{w_i(x), H_{-j-1}\}_1, \quad j = 1, 2, 3, \dots. \quad (82)$$

The flows of the bihamiltonian hierarchy are generated by the Hamiltonians  $H_{-j}, j < 0$  via:

$$\frac{\partial w_i}{\partial t_{-j+2}} = \{w_i(x), H_{-j}\}_1, \quad j = 3, \dots, i = 1, 2. \quad (83)$$

The lower Hamiltonians  $H_{-j}$  for  $j > 1$  can be obtained recursively from the Casimir  $H_{-1} = \int w_2(x)dx$  of the first Poisson bracket applying the Lenard relations (82). Following [3, 4], we introduce objects  $\varphi_1, \varphi_2$  defined by  $w_1 = \varphi_1 - \varphi_{1,x}/\mu$ ,  $w_2 = \varphi_2 + \varphi_{2,x}/\mu$ . Then

$$\{\varphi_1(x), w_2(x')\}_1 = \delta'(x - x'), \quad \{w_1(x), \varphi_2(x')\}_1 = \delta'(x - x')$$

and the Lenard relations yield

$$\begin{aligned} H_{-2} &= \int [\varphi_2(\varphi_1 - \varphi_{1,x}/\mu)] dx \\ H_{-3} &= \frac{1}{2} \int [\varphi_2^2 + \varphi_2 \varphi_1 (\varphi_1 - \varphi_{1,x}/\mu)] dx. \end{aligned}$$

Plugging the above  $H_{-3}$  into equation of motion (83) for  $j = 3$  we obtain:

$$\begin{aligned}(w_1)_t &= \left( \varphi_2 + \frac{1}{2}\varphi_1^2 - \frac{1}{2\mu}\varphi_1\varphi_{1,x} \right)_x, \\ (w_2)_t &= \left( \varphi_1\varphi_2 + \frac{1}{2\mu}\varphi_1\varphi_{2,x} \right)_x,\end{aligned}\tag{84}$$

where  $t = t_{-1}$ . Defining  $u$  such that  $\varphi_1 = 2u$  and  $\rho$  such that  $w_2 = -\rho^2/\mu^2 + w_1^2/4$  or  $\rho^2 = w_1^2\mu^2/4 - w_2\mu^2$  we can rewrite the above system of equations after a transformation  $t \rightarrow -t$  as

$$u_t - u_{xxt}/\mu^2 = \rho\rho_x/\mu^2 - 3uu_x + 2u_xu_{xx}/\mu^2 + uu_{xxx}/\mu^2\tag{85}$$

$$\rho_t = -(u\rho)_x,\tag{86}$$

which agrees with the 2-component Camassa-Holm equation. Multiplying equation (85) by  $\mu^2$  and taking  $\mu^2 \rightarrow 0$  yields

$$-u_{xxt} = \rho\rho_x + 2u_xu_{xx} + uu_{xxx}\tag{87}$$

corresponding to eq. (26) with  $\kappa = 0$ .

In order to take the  $\mu \rightarrow 0$  limit of the Poisson structure (81) it is convenient to change the variables from  $w_1, w_2$  to  $m$  and  $\rho$  defined as

$$\begin{aligned}m &= \frac{1}{2}\mu^2(w_1(x) + w_{1,x}/\mu) = \mu^2u - u_{xx} \\ \rho^2 &= \mu^2(w_1^2/4 - w_2).\end{aligned}\tag{88}$$

In terms of  $m$  and  $\rho$  the Poisson bracket structure (81) turns into:

$$\begin{aligned}\{m(x), m(x')\}_1 &= 0, \\ \{\rho^2(x), \rho^2(x')\}_1 &= -\mu^2(2m(x)\delta'(x-y) + m'(x)\delta(x-y)), \\ \{\rho^2(x), m(x')\}_1 &= \frac{1}{2}\mu^2(-\mu^2\delta'(x-x') + \delta'''(x-x')), \\ \{m(x), m(x')\}_2 &= \frac{1}{2}\mu^2(\mu^2\delta'(x-x') - \delta'''(x-x')), \\ \{\rho(x), \rho(x')\}_2 &= -\frac{1}{2}\mu^2\delta'(x-x'), \\ \{\rho(x), m(x')\}_2 &= 0,\end{aligned}\tag{89}$$

## 5.2 The $\mu \rightarrow 0$ limit and the Dym type hierarchy.

Redefining the brackets as follows

$$\{\cdot, \cdot\}_j \longrightarrow \mu^2\{\cdot, \cdot\}_j$$

and taking  $\mu \rightarrow 0$  limit in equation (89) we find for the first and second bracket structure in terms of  $u$  and  $\rho$  (see also [15, 19]):

$$\begin{aligned} \{u(x), u(x')\}_1 &= 0, \\ \{\rho^2(x), \rho^2(x')\}_1 &= 2u''(x')\delta'(x - x') - u'''(x')\delta(x - x') \\ \{\rho^2(x), u(x')\}_1 &= -\frac{1}{2}\delta'(x - x'), \\ \{u(x), u(x')\}_2 &= -\frac{1}{2}\partial_x^{-1}\delta(x - x'), \\ \{\rho(x), \rho(x')\}_2 &= -\frac{1}{2}\partial_x\delta(x - x'), \\ \{u(x), \rho(x')\}_2 &= 0, \end{aligned} \tag{90}$$

The first bracket in (90) has the Casimir:

$$H_{-1}^{(1)} = \int [\rho^2(x) - u_x^2(x)] dx.$$

This Casimir leads via Lenard relation (82) to the Hamiltonian:

$$H_{-2} = 2 \int (\rho^2 - u_x^2) u dx,$$

which in turn generates equations of motion of  $\mu = 0$  case via equations (83) and (82):

$$\begin{aligned} \frac{\partial u_x}{\partial t} &= \frac{1}{2}\{u_x(x), H_{-2}\}_2 = -\frac{1}{2}(u_x)^2 - uu_{xx} - \frac{1}{2}\rho^2 \\ \frac{\partial \rho}{\partial t} &= \frac{1}{2}\{\rho(x), H_{-2}\}_2 = -(u\rho)_x \end{aligned} \tag{91}$$

This Hamiltonian structure can be extended by an additional term:

$$\bar{H}_{-2} = -2\kappa \int u^2 dx.$$

Adding this term to  $H_{-2}$  will lead via relations (91) to correct equations of motion (29)-(30).

### 5.3 Hamiltonians of positive order

There exists another class of conserved charges, different from the chain of Hamiltonians  $H_{-j}$ ,  $j = 1, 2, \dots$  of negative order discussed above. These are the Hamiltonians of positive order originating from the Casimir

$$H_{-1}^{(2)} = 2 \int \rho(x) dx \tag{92}$$

of the second Poisson bracket (89). We now employ Lenard relations (82) to construct higher order Hamiltonians. The first recurrence step:

$$\{\cdot, \int \rho(x) dx\}_1 = \{\cdot, H_0\}_2,$$

where “.” stands for phase space variables  $m(x)$  and  $\rho(x)$ , leads to a new Hamiltonian:

$$H_0 = - \int \frac{m(x)}{\rho(x)} dx . \quad (93)$$

in agreement with expression found in [4]. The integrand of  $H_0$  can be rewritten as

$$\frac{m}{\rho} = \rho \frac{m}{\rho^2} = \rho P = \rho f_y = f_x$$

and therefore  $H_0$  appears to be a surface term that would vanish if  $f$  would be a local field.

On the next level we find from the Lenard relations:

$$\{\cdot, H_0\}_1 = \{\cdot, H_1\}_2 ,$$

with

$$H_1 = \int \left[ \frac{1}{4\rho^3} (\rho_x^2 - m^2) + \frac{\mu^2}{4\rho} \right] dx \quad (94)$$

This recurrence process can be continued to yield higher order Hamiltonians. Technical calculations involved in obtaining higher order Hamiltonians become increasingly tedious. Remarkably, we can bypass these difficulties by relying on the underlying AKNS structure governing higher positive flows. We recall the Hamiltonian densities  $\mathcal{H}_n$  (8) of the AKNS model generating the positive flows of the model. Their conservation law with respect to the negative flow  $s$  takes a form

$$(\mathcal{H}_n)_s = X_y , \quad (95)$$

where  $s$  and  $y$  are “reciprocal” variables describing time and space of the AKNS model and  $X$  is some local quantity. The above relation ensures that  $(\int \mathcal{H}_n dy)_s = 0$  (for  $X$  local in  $u$  and  $\rho$ ) and thus the integral  $\int \mathcal{H}_n dy$  is conserved.

In terms of the original  $t, x$  variables the conservation laws (95) read

$$\left( \frac{\partial}{\partial t} + \rho u \frac{\partial}{\partial y} \right) \mathcal{H}_n = \frac{\partial}{\partial y} X$$

or

$$\frac{\partial}{\partial t} \mathcal{H}_n = -\rho u \frac{\partial}{\partial y} \mathcal{H}_n + \frac{\partial}{\partial y} X = -u \frac{\partial}{\partial x} \mathcal{H}_n + \frac{1}{\rho} \frac{\partial}{\partial x} X ,$$

where we used that  $\partial/\partial y = \rho \partial/\partial x$ . It follows that

$$\frac{\partial}{\partial t} (\rho \mathcal{H}_n) = -(u\rho)_x \mathcal{H}_n - u\rho \mathcal{H}_{nx} + X_x = \frac{\partial}{\partial x} (X - u\rho H_n) . \quad (96)$$

Thus, the quantities  $\rho \mathcal{H}_n$  are conserved charges of the 2-component Camassa-Holm and 2-component Dym type models.

The first two Hamiltonian densities  $rq$  and  $rq_y$  of the AKNS model, given by relations (9) and (10), give rise, after use of definitions (47),(36) and (40), to the following conserved

charges:

$$\begin{aligned}
H_1 &= \int \rho \mathcal{H}_1 dx = \int \rho r q dx = \int \left[ \frac{1}{4\rho^3} (\rho_x^2 - m^2) + \frac{\mu^2}{4\rho} \right] dx, \\
H_2 &= \frac{1}{2} \int \rho \mathcal{H}_2 dx = \frac{1}{2} \int \rho r q_y dx = \frac{1}{2} \int r q_x dx \\
&= \int \frac{m}{2\rho^2} \left[ \frac{\mu^2}{4\rho} - \frac{3\rho_x^2}{4\rho^3} + \frac{\rho_{xx}}{2\rho^2} - \frac{m^2}{4\rho^3} \right] dx,
\end{aligned} \tag{97}$$

which we have verified explicitly to be conserved under equations of motion (19)-(20) for  $m = \mu^2 u - u_{xx}$  for  $\mu \neq 0$  and  $m = -u_{xx} + \kappa/2$  for  $\mu = 0$ .

We recognize in  $\rho \mathcal{H}_1$  the Hamiltonian  $H_1$  derived in (94) from the Casimir of the second bracket via Lenard recursion relations. Furthermore, we have shown that  $H_1 = \int \rho \mathcal{H}_1 dx$  and  $H_2 = \int \rho \mathcal{H}_2 dx / 2$  also are interrelated via the Lenard relation:

$$\{\cdot, H_1\}_1 = \{\cdot, H_2\}_2.$$

Therefore, the conclusion is that the AKNS induced Hamiltonians  $\rho \mathcal{H}_n$  form the sequence of positive order Hamiltonians of the 2-component Camassa-Holm and 2-component Dym type hierarchies. The formula (8) given in section 2 can be used to systematically derive all the Hamiltonians governing positive flows of this model.

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## References

- [1] R. Camassa and D.D. Holm, Phys. Rev. Lett. **71** 1661 (1993)
- [2] M. Chen, S-Q. Liu, Y. Zhang, A 2-component generalization of the Camassa-Holm equation and its solutions, [nlin.SI/0501028]
- [3] S.Q. Liu, Y. Zhang, Jour. Geom. Phys. **54**, 427 (2005) [math-dg/0405146]
- [4] G. Falqui, On a Camassa-Holm type equation with two dependent variables, [nlin.SI/0505059]
- [5] M.D. Kruskal, Nonlinear wave equations. In: Moser, J. (eds.) Dynamical Systems, Theory and Applications. Lecture Notes in Physics 38, New York: Springer (1975)
- [6] C. Cewen, Acta Math. Sin. **6**, 35 (1990)
- [7] J. Hunter, Y. Zheng, Physica **D 79**, 361 (1994)

- [8] M.S. Alber, R. Camassa, Yu. Fedorov, D.D. Holm and J.E. Marsden, Phys. Lett. **A** **264**, 171 (1999)
- [9] M.S. Alber, R. Camassa, D.D. Holm and J.E. Marsden, PDEs, Proc. Roy. Soc. **450**, 677 (1995)
- [10] J. C. Brunelli, A. Das and Z. Popowicz, J. Math. Phys. **45**, 2646 (2004) [arXiv:nlin.si/0307043].
- [11] H. Aratyn, J.F. Gomes, E. Nissimov, S. Pacheva and A.H. Zimerman, “*Symmetry Flows, Conservation Laws and Dressing Approach to Integrable Models*” in Integrable Hierarchies and Modern Physical Theories, Ed. H. Aratyn and A. Sorin, Kluwer Academic Publishers (2001), 243-275
- [12] H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, J. Phys. A **33**, L331 (2000) [nlin.si/0007002]
- [13] H. Aratyn, J. F. Gomes and A. H. Zimerman, Nucl. Phys. B **676**, 537 (2004) [hep-th/0309099]
- [14] H. Aratyn, J. F. Gomes and A. H. Zimerman, J. Geom. Phys. **46**, 21 (2003) [Erratum-ibid. **46**, 201 (2003)] [hep-th/0107056]
- [15] M. V. Pavlov, J. Phys. A: Math. Gen. **38**, 3823 (2005) [nlin.SI/0412072]
- [16] P. Rosenau, Phys. Rev. Lett. **73**, 1737 (1994)
- [17] H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, J. Phys. A **31**, 9483 (1998) [solv-int/9709004]; I. Cabrera-Carnero, J. F. Gomes, E. P. Gueuvoghlianian, G. M. Sotkov and A. H. Zimerman, [hep-th/0109117], contribution to 7-th International Wigner Symposium (Wigsym 7), 24-29 Aug 2001, College Park, Maryland
- [18] R. Camassa and A.I. Zenchuk, Phys. Lett. **A** **281**, 26 (2001); R.A. Kraenkel and A.I. Zenchuk, J. Phys. A: Math. Gen. **32**, 4733 (1999)
- [19] B.A. Dubrovin, S.P. Novikov, Hydrodynamics of soliton lattices. Soviet Scientific Reviews, Section C: Mathematical Physics Reviews 9 Part 4. Harwood Academic Publishers GmbH, Yverdon (1993) 136 pp., as mentioned in [15].